Chapter 5 and 6

Section 1: Vector Spaces

(chapter 4 video part 2 & book section 6.1)

Ideas in this section…

1) Definition of a Vector Space
2) Example of Vector Spaces
3) Weird Examples/Non-examples of Vector Spaces
4) Some Results About Vector Spaces

Definition of a Vector Space

Def: A vector space consists of…

1) a set of objects V
   - objects in V are called vectors
   - notation for objects in V will be written with an arrow above it (like $\vec{v}$)
2) a set of numbers
   - these numbers are called scalars
   - for us, this set will be $\mathbb{R}$
   - notation for scalars will not have an arrow above it (like c)
3) a rule for taking any 2 vectors and producing a 3rd vector
   - this is called vector addition
   - if $\vec{v}, \vec{w} \in V$, the notation for the new vector is $\vec{v} + \vec{w}$
4) a rule for taking a scalar and a vector and producing another vector
   - this is called scalar multiplication
   - if $c \in \mathbb{R}$ and $\vec{v} \in V$, the notation for the new vector is $c\vec{v}$

where the following 10 properties are satisfied…

Definition of a Vector Space

Def: A vector space consists of…

where the following 10 properties are satisfied…

Axioms for vector addition

A1. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are in* V, *then* $\mathbf{u}+\mathbf{v}$ *is in* V.

A2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ *for all* $\mathbf{u}$ *and* $\mathbf{v}$ *in* V.

A3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$ *for all* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *in* V.

A4. *An element* $\mathbf{0}$ *in* V *exists such that* $\mathbf{v}+\mathbf{0}=\mathbf{v}=\mathbf{0}+\mathbf{v}$ *for every* $\mathbf{v}$ *in* V.

A5. *For each* $\mathbf{v}$ *in* V, *an element* $-\mathbf{v}$ *in* V *exists such that* $-\mathbf{v}+\mathbf{v}=\mathbf{0}$ *and* $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$.

Axioms for scalar multiplication

S1. *If* $\mathbf{v}$ *is in* V, *then* $a\mathbf{v}$ *is in* V *for all* a *in* $\mathbb{R}$.

S2. $a(\mathbf{v}+\mathbf{w})=a\mathbf{v}+a\mathbf{w}$ *for all* $\mathbf{v}$ *and* $\mathbf{w}$ *in* V *and all* a *in* $\mathbb{R}$.

S3. $(a+b)\mathbf{v}=a\mathbf{v}+b\mathbf{v}$ *for all* $\mathbf{v}$ *in* V *and all* a *and* b *in* $\mathbb{R}$.

S4. $a(b\mathbf{v})=(ab)\mathbf{v}$ *for all* $\mathbf{v}$ *in* V *and all* a *and* b *in* $\mathbb{R}$.

S5. $1\mathbf{v}=\mathbf{v}$ *for all* $\mathbf{v}$ *in* V.

Examples Of Vector Spaces

Ex 1: $\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R} \}$ with the usual addition and scalar multiplication is a vector space over $\mathbb{R}$.

Addition:

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) \equiv (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

Scalar Multiplication:

$$c(x_1, x_2, \ldots, x_n) \equiv (cx_1, cx_2, \ldots, cx_n)$$

Examples Of Vector Spaces

Ex 2: $M_{mn}=$ the set of all $m \times n$ with entries in $\mathbb{R}$ with the usual rule for matrix multiplication and scalar multiplication is a vector space over $\mathbb{R}$.

Addition:
Add corresponding components

Scalar Multiplication:
To multiply a matrix by a scalar, multiply each component of the matrix by that scalar

Examples Of Vector Spaces

Ex 3: P $=$ the set of all polynomials with real coefficients is a vector space over $\mathbb{R}$ with the following rules for polynomial addition and scalar multiplication

Addition: If $p(x), q(x) \in P$, k is the larger of the degrees of $p(x)$ and $q(x)$,
$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$ and
$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_k x^k$, then

$$p(x) + q(x) \equiv (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_k + b_k)x^k$$

Scalar Multiplication:
If $p(x) \in P$, $c \in \mathbb{R}$, and $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$, then

$$cp(x) \equiv (ca_0) + (ca_1)x + (ca_2)x^2 + \cdots + (ca_m)x^m$$

Examples Of Vector Spaces

Ex 4: $\mathrm{P_n} =$ the set of all polynomials of degree $\leq n$ with real coefficients is a vector space over $\mathbb{R}$ with the following rules for polynomial addition and scalar multiplication

Addition: If $p(x), q(x) \in P_n$, and $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and $q(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$, then

$$p(x) + q(x) \equiv (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n$$

Scalar Multiplication:
If $p(x) \in P_n, c \in \mathbb{R}$, and $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, then

$$cp(x) \equiv (ca_0) + (ca_1)x + (ca_2)x^2 + \cdots + (ca_n)x^n$$

Examples of Vector Spaces

<u>Ex 5</u>: $F[a, b] =$ the set of all real-valued functions defined on $[a, b]$ is a vector space over $\mathbb{R}$ with the following rules for function addition and scalar multiplication

<u>Addition</u>: If $f, g \in F[a, b]$, then

$$(f + g)(x) \equiv f(x) + g(x)$$

<u>Scalar Multiplication</u>:
If $f \in F[a, b]$, and $c \in \mathbb{R}$, then

$$(cf)(x) \equiv cf(x)$$

Weird Examples / Nonexamples of Vector Spaces

Ex 6:

Example 6.1.4

Let V denote the set of all ordered pairs (x, y) and define addition in V as in $\mathbb{R}^2$. However, define a new scalar multiplication in V by

$$a(x, y) = (ay, ax)$$

Determine if V is a vector space with these operations.

Weird Examples / Nonexamples of Vector Spaces

Ex 7:

Exercise 6.1.4 If V is the set of ordered pairs $(x,\ y)$ of real numbers, show that it is a vector space with addition $(x,\ y)+(x_1,\ y_1)=(x+x_1,\ y+y_1+1)$ and scalar multiplication $a(x,\ y)=(ax,\ ay+a-1)$. What is the zero vector in V?

Weird Examples / Nonexamples of Vector Spaces

Ex 8:

Exercise 6.1.3 Let V be the set of positive real numbers with vector addition being ordinary multiplication, and scalar multiplication being $a \cdot v = v^a$. Show that V is a vector space.

Some Results About Vector Spaces

Theorem 6.1.1: Cancellation

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in a vector space V. If $\mathbf{v}+\mathbf{u}=\mathbf{v}+\mathbf{w}$, then $\mathbf{u}=\mathbf{w}$.

Proof:

Some Results About Vector Spaces

<u>Theorem:</u>

1. The zero vector in a vector space is unique. That is, if $\vec{0}_1, \vec{0}_2 \in V$ are both zero vectors, then $\vec{0}_1 = \vec{0}_2$.
2. Given a vector $\vec{v} \in V$, its additive inverse $-\vec{v}$ is unique. That is, if $\vec{a}, \vec{b} \in V$ are both additive inverses of $\vec{v}$, then $\vec{a} = \vec{b}$.

<u>Proof of 1:</u>

Some Results About Vector Spaces

Theorem:

1. The zero vector in a vector space is unique. That is, if $\vec{0}_1, \vec{0}_2 \in V$ are both zero vectors, then $\vec{0}_1 = \vec{0}_2$.
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Proof of 2:

Some Results About Vector Spaces

Theorem 6.1.3

Let $\mathbf{v}$ *denote a vector in a vector space* V *and let* a *denote a real number.*

1. $0\mathbf{v} = \mathbf{0}$.
2. $a\mathbf{0} = \mathbf{0}$.
3. *If* $a\mathbf{v} = \mathbf{0}$, *then either* $a = 0$ *or* $\mathbf{v} = \mathbf{0}$.
4. $(-1)\mathbf{v} = -\mathbf{v}$.
5. $(-a)\mathbf{v} = -(a\mathbf{v}) = a(-\mathbf{v})$.

Proof of 1:

Some Results About Vector Spaces

Theorem 6.1.3

Let $\mathbf{v}$ denote a vector in a vector space V and let a denote a real number.

1. $0\mathbf{v} = \mathbf{0}$.
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4. $(-1)\mathbf{v} = -\mathbf{v}$.
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<u>Proof of 2:</u>

Some Results About Vector Spaces

Theorem 6.1.3

Let $\mathbf{v}$ *denote a vector in a vector space* V *and let* a *denote a real number.*

1. $0\mathbf{v} = \mathbf{0}$.
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4. $(-1)\mathbf{v} = -\mathbf{v}$.
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Proof of 3:

Some Results About Vector Spaces

Theorem 6.1.3

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4. $(-1)\mathbf{v} = -\mathbf{v}$.
5. $(-a)\mathbf{v} = -(a\mathbf{v}) = a(-\mathbf{v})$.

<u>Proof of 4:</u>

Some Results About Vector Spaces

Theorem 6.1.3

Let $\mathbf{v}$ *denote a vector in a vector space* V *and let* a *denote a real number.*

1. $0\mathbf{v} = \mathbf{0}$.
2. $a\mathbf{0} = \mathbf{0}$.
3. *If* $a\mathbf{v} = \mathbf{0}$, *then either* $a = 0$ *or* $\mathbf{v} = \mathbf{0}$.
4. $(-1)\mathbf{v} = -\mathbf{v}$.
5. $(-a)\mathbf{v} = -(a\mathbf{v}) = a(-\mathbf{v})$.

Proof of 5:

What you need to know from the book

Book reading

Pages: 329 – 336 all

Problems you need to know how to do from the book

#'s 1-4, 6-12, 14-17